On the Difference of Energy between the Einstein and Møller Prescription

I-Ching Yang[†] [1] and Irina Radinschi[‡] [2]

[†]Department of Natural Science Education and
Advanced Science and Technology Research Center,
National Taitung Teachers College, Taitung, Taiwan 950, Republic of China
and
[‡]Department of Physics, "Gh. Asachi" Technical University,
Iasi, 6600, Romania

ABSTRACT

In some black hole solutions, these do not exist the same energy-momentum complexes associated with using definition of Einstein and Møller in given coordinates. Here, we consider the difference of energy between the Einstein and Møller prescription, and compare it with the energy density of those black hole solutions. We found out a special relation between the difference of energy between the Einstein and Møller prescription and the energy density for considered black hole solutions.

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[1]E-mail:icyang@cc.ntttc.edu.tw
[2]E-mail:iradinsc@phys.tuiasi.ro

In the theory of general relativity, many physicists, like Einstein [1], Landau and Lifshitz [2], Tolman [3], Papapetrou [4], Møller [5], and Weinberg [6], had given different definitions for the energy-momentum complex. Specifically, the Møller energy-momentum complex allows to compute the energy in any spatial coordinate system. Some results recently obtained [7, 8, 9, 10] sustain that the Møller energy-momentum complex is a good tool for obtaining the energy distribution in a given space-time. Also, in his recent paper, Lessner [11] gave his opinion that the Møller definition is a powerful concept of energy and momentum in general relativity. In his paper Virbhadra [12] point out that several energy-momentum complexes (ELLPW) give the same result for a general non-static spherically symmetric space-time of the Kerr-Schild class.

In particular, whatever coordinates do not exist the same energy complexes associated with using definitions of Einstein and Møller in some space-time solutions [7, 13]. According to the definition, the Einstein energy complex is [1]

$$E_{\mathrm{Ein}} = \frac{1}{16\pi}\int \frac{\partial {H_0}^{0l}}{\partial x^l} d^3x, \tag{1}$$

where

$$ {H_0}^{0l} = \frac{g_{00}}{\sqrt{-g}} \frac{\partial}{\partial x^m}[(-g)g^{00}g^{lm}], \tag{2}$$

and the Møller energy complex is [5]

$$E_{\mathrm{Møl}} = \frac{1}{8\pi}\int \frac{\partial {\chi_0}^{0l}}{\partial x^l} d^3x, \tag{3}$$

where

$$ {\chi_0}^{0l} = \sqrt{-g} g^{0\beta} g^{l\alpha} \left(\frac{\partial g_{0\alpha}}{\partial x^\beta} - \frac{\partial g_{0\beta}}{\partial x^\alpha} \right). \tag{4}$$

Where the Latin indices take values from 1 to 3, and the Greek indices run from 0 to 3. Let us look into the difference of energy between the Einstein and Møller prescription, which be defined as

$$\Delta E = E_{\mathrm{Ein}} - E_{\mathrm{Møl}} \tag{5}$$

In this article, we would discuss the problem within the difference between Einstein and Møller energy-momentum complexes.

In the first case, we think of two solutions of Einstein vacuum field equation:

(i) Schwarzschild space-time

The metric form of Schwarzschild space-time is

$$ds^2 = f dt^2 - f^{-1} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\varphi^2, \tag{6}$$

where $f = 1 - 2M/r$. It is a well-known results that the energy complexes of Einstein and Møller of Schwarzschild space-time are

$$E_{\mathrm{Ein}} = M, \tag{7}$$

$$E_{\mathrm{Møl}} = M, \tag{8}$$

and the difference is

$$\Delta E = 0. \tag{9}$$

(ii) Kerr solution

The metric form of Kerr solution is considered as

$$ds^2 = \alpha dt^2 - \beta dr^2 - \gamma d\theta^2 - \delta d\phi^2 - 2\sigma dt d\varphi, \tag{10}$$

where $\alpha = 1 - 2Mr/\Sigma$, $\beta = \Sigma/\Delta$, $\gamma = \Sigma$, $\delta = r^2 + a^2 + 2Ma^2 r \sin^2\theta/\Sigma$ and $\sigma = 2Mar\sin^2\theta/\Sigma$. Here $\Sigma \equiv r^2 + a^2\cos^2\theta$ and $\Delta \equiv r^2 - 2Mr + a^2$. To use the results in the Virbhadra articles [14] and to set these $Q = 0$, we could obtain the enrgy-momentum complexes of Einstein and Møller of Kerr space-time are

$$E_{\mathrm{Ein}} = M, \tag{11}$$

$$E_{\mathrm{Møl}} = M, \tag{12}$$

and the difference is

$$\Delta E = 0. \tag{13}$$

For Einstein's vacuum field equation, the energy density is

$$T_0{}^0 = 0. \tag{14}$$

We would find that ΔE equal to the value of $T_0{}^0$.

Next, we consider two case of the coupled system of the Einstein field and electromagnetic field:

(iii) Reissner-Nordström space-time

The metric form of Reissner-Nordström space-time is

$$ds^2 = f dt^2 - f^{-1} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\varphi^2, \tag{15}$$

where $f = 1 - 2M/r + Q^2/r^2$. Previously, the energy-momentum complexes of Einstein and Møller of Reissner-Nordström space-time had been calculated with

$$E_{\mathrm{Ein}} = M - \frac{Q^2}{2r}, \tag{16}$$

$$E_{\mathrm{Møl}} = M - \frac{Q^2}{r}, \tag{17}$$

and the difference is

$$\Delta E = \frac{Q^2}{2r}. \tag{18}$$

Notice that the energy density in the Einstein-Maxwell field equation is

$$T_0{}^0 = \frac{Q^2}{r^4}. \tag{19}$$

(iv) charged regular black hole

The metric form of charged regular black hole is [15]

$$ds^2 = f dt^2 - f^{-1} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\varphi^2, \tag{20}$$

where $f = 1 - \frac{2M}{r}(1 - \tanh(\frac{Q^2}{2Mr}))$. Using the results of Radinschi's articles [10], the energy-momentum complexes of Einstein and Møller of charged regular black hole are

$$E_{\mathrm{Ein}} = M\left[1 - \tanh(\frac{Q^2}{2Mr})\right], \tag{21}$$

$$E_{\mathrm{Møl}} = M\left[1 - \tanh(\frac{Q^2}{2Mr})\right] - \frac{Q^2}{2r}\left[1 - \tanh^2(\frac{Q^2}{2Mr})\right], \tag{22}$$

and the difference is

$$\Delta E = \frac{Q^2}{2r}\left[1 - \tanh^2(\frac{Q^2}{2Mr})\right]. \tag{23}$$

However, the energy density of the coupled system of the Einstein field and nonlinear electrodynamics field is

$$T_0{}^0 = \frac{Q^2}{r^4}\left[1 - \tanh^2(\frac{Q^2}{2Mr})\right]. \tag{24}$$

Here the relation between ΔE and the energy density is written as

$$\Delta E = T_0{}^0 \times (\frac{r^3}{2}). \tag{25}$$

Then, in the last case, we use a special black hole solution without singularity:

(v) static spherically symmetric nonsingular black hole

The metric form of the static spherically symmetric nonsingular black hole solution [16] is

$$ds^2 = f dt^2 - f^{-1} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2, \tag{26}$$

where $f = 1 - R_g/r$ and $R_g = r_g(1 - \exp(r^3/r_*^3))$, $r_*^3 = r_g r_0^2$,$r_g = 2M$, $r_0^2 = \frac{3}{8\pi\varepsilon_0}$. According to the results of our articles [17], the energy-momentum complexes of Einstein and Møller of the static spherically symmetric nonsingular black hole are

$$\begin{aligned} E_{\text{Ein}} &= M - M\exp(-\frac{r^3}{r_*^3}), & (27) \\ E_{\text{Møl}} &= M - M\exp(-\frac{r^3}{r_*^3}) - \frac{3r^3}{r_0^2}\exp(-\frac{r^3}{r_*^3}), & (28) \end{aligned}$$

and the difference is

$$\Delta E = \frac{3r^3}{r_0^2}\exp(-\frac{r^3}{r_*^3}). \tag{29}$$

Notice that the energy density of the static spherically symmetric nonsingular black hole be assumed as

$$T_0{}^0 = \frac{3}{r_0^2}\exp(-\frac{r^3}{r_*^3}). \tag{30}$$

The relation between ΔE and the energy density is written as

$$\Delta E = T_0{}^0 \times r^3. \tag{31}$$

Although, we could summarize that the general relation between ΔE and the energy density $T_0{}^0$ be written as

$$\Delta E = T_0{}^0 \times (kr^3), \tag{32}$$

with $k = 1/2$ and $k = 1$. But, it is still an open question why the special relation has between ΔE and the energy density $T_0{}^0$. Further study is needed to understand the difference between the Einstein and Møller energy complexes of more varied black hole solutions.

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